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2005 J. Phys. A: Math. Gen. 38 6893

(http://iopscience.iop.org/0305-4470/38/31/003)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6893–6915 doi:10.1088/0305-4470/38/31/003

Quantum dynamical entropies for discrete classical systems: a comparison

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Received 10 March 2005, in final form 6 June 2005
Published 20 July 2005
Online at stacks.iop.org/JPhysA/38/6893

Abstract

On a family of classical dynamical systems on the 2-torus, we perform a discretization procedure similar to the anti-Wick quantization. Such a discretization is performed by using a particular class of states, fulfilling an appropriate dynamical localization property, typical of quantum coherent states. The same set of states is involved in the construction of a quantum entropy, that we test on the discrete approximants; a correspondence with the classical metric entropy of Kolmogorov–Sinai is found only over time scales that are logarithmic in the discretization parameter.

PACS numbers: 05.45.Ac, 05.45.Mt, 03.65.Fd, 45.05.+x
Mathematics Subject Classification: 37D20, 54C70, 28D20, 81Q20, 81R30

1. Introduction

Under the term *classical chaos* is included a rich phenomenology of classical dynamical systems on a compact phase space characterized by a high sensitivity to initial conditions: if very small initial errors exponentially amplify during the temporal evolution, the systems is called *chaotic* [1–7]. Nevertheless, the motion being confined within a bounded region, the exponential divergence of trajectories has to be tested in a finite domain. This leads to defining the (maximal) coefficient of such exponential amplification, which is called the *Lyapunov exponent*, as $\xi := \lim_{n\to\infty}(1/n)\lim_{\delta\to 0}\log(\delta_n/\delta)$, where we consider the initial error δ growing as δ_n under a discrete time evolution. When the amplification of errors is exponential, the Lyapunov exponent ξ is positive and the system is classified as chaotic. $\xi = 0$ is typical of regular time evolutions, but this also happens if we forbid δ to go to zero; indeed, $\delta_n \leqslant \Delta$ and $\lim \frac{1}{n}$ vanishes. This occurs for instance in the case of quantum dynamical systems, where the uncertainty principle naturally endows the phase space with an $\hbar$-dependent granularity, and the $\delta \to 0$ limit cannot be achieved for finite $\hbar > 0$, but only if we perform the classical limit $\hbar \to 0$ before the time one. Although this shows the non-commutativity of the classical and the time limits [2, 6], the temporal evolution of a finite-dimensional quantization compared with its classical counterpart exhibits good agreement on a time scale bounded by the so-called *breaking time* $\tau_{\mathrm{B}}(\hbar)$: usually, when the classical system is chaotic, τ_{B} scales logarithmically in $\hbar$ [1, 2, 6, 8–10], whereas for regular systems the scaling is $\hbar^{-\alpha}$ for some $\alpha > 0$ [1].

A similar phenomenon can be observed in discrete classical systems, that are obtained for instance by forcing a classical system to live on a square lattice of N^2 points, whose minimal spacing $a = \frac{1}{N}$ acts as a lower bound for $\delta \to 0$: in this case, $\frac{1}{N}$ plays in the discrete domain the same role that $\hbar$ plays in the quantum one and can be interpreted as a quantization-like parameter.

By using this analogy of behaviour between quantum and discrete classical systems, the study of the latter result is quite interesting and promising; indeed we can get all the benefits arising from classicality, that is the simplicity due to commutativity, and deeply enquire into the chaotic property in these kinds of ‘toy models’.

Since finite-dimensional quantizations of classical dynamical systems have an algebraic formulation, this can be easily extended to *discretization procedures* when we restrict from the full matrix algebra of bounded operators on a Hilbert space, typical of quantum systems, to a commutative algebra of diagonal operators describing a classical system [11].

A very useful tool of the semiclassical analysis of quantum systems is represented by the use of coherent states and a standard quantization scheme, the anti-Wick one [12], is based on them: by mimicking this procedure we set up a discretization involving a class of states that we will refer to as lattice states, suitably defined on our Hilbert space. Of course, in order to have a good quantization, the *classical limit* $\hbar \to 0$ has to be tested [13] and a large part of this work has been devoted to giving and proving a consistent definition of a *continuous limit* $N \to \infty$, suited for a reasonable algebraic discretization scheme.

A first result in this direction is that the convergence of the discrete to the continuous dynamics is due to a very special property of lattice states, that is known as the *dynamical localization* property [14].

We apply our discretization procedure to a well-known class of classical systems [7] that are represented by integer matrix action on the 2-torus; such systems can be rigorously divided into three families, namely *hyperbolic*, *parabolic* and *elliptic*, characterized by different chaotic properties. As expected, differences in the behaviour of the breaking times $\tau_{\mathrm{B}}(N)$ (now of discrete/continuous correspondence) are found in the three different regimes.

The Lyapunov exponent is zero on systems with a finite number of states (both discrete and quantum) because it is an asymptotic quantity: an alternative approach is to enquire into the chaotic properties of a system during its temporal evolution, and whether the system exhibits some kind of *finite-time chaos*. For classical dynamical systems the Pesin–Ruelle theorem [15] establishes a bridge between chaos and information, giving a relation between the Kolmogorov–Sinai *metric* entropy and the sum of all positive Lyapunov exponents. Moreover, although the metric entropy is defined as a (partial) entropy production in the long run [7, 16], such a partial entropy can be observed and analysed even during the temporal evolution, that is at finite times.

With the aim of using entropy to detect chaos, several *quantum dynamical entropies* have been introduced. In a recent work [14], two of them, called CNT (Connes, Narnhofer and Thirring) [17] and ALF (Alicky, Lindblad and Fannes) [18] are shown to converge to the KS invariant (but only in a joint time and classical limit) when applied to the anti-Wick quantization of the hyperbolic family of the classical dynamical systems mentioned above. Only the hypothesis of dynamical localization for coherent states was used in obtaining that result. Instead of extending such a result to our discretization scheme, we directly study another quantum dynamical entropy, constructed by means of coherent states and so-called CS-quantum entropy [19].

What we show is that the CS-entropy production of a discrete classical system does converge to the KS-entropy production of the continuous limit, but only over time scales logarithmic in the quantization-like parameter $\frac{1}{N}$. This confirms the numerical results obtained in [20] for the ALF-entropy on a similar class of discrete systems, but within the Weyl quantization-like scheme instead of the anti-Wick.

Finally, we divided the CS-quantum entropy into its dynamical and measure-dependent parts, and we show the latter does not play a role in the (positive) entropy rate.

2. Classical dynamical systems and phase-space discretization

The typical description of a classical dynamical system is given by means of a measure space $\mathcal{X}$, the phase space, endowed with the Borel σ-algebra of its measurable subsets and a normalized measure μ ($\mu(\mathcal{X}) = 1$). The probability that phase points belong to measurable subsets $E \subseteq \mathcal{X}$ is given by the 'volumes' $\mu(E) = \int_E \mu(\mathrm{d}\boldsymbol{x})$; so the measure μ defines the statistical properties of the system and represents a possible 'state'.

Every reversible discrete time dynamics amounts to an invertible measurable map $T : \mathcal{X} \mapsto \mathcal{X}$ such that $\mu \circ T = \mu$, and to its iterates $\{T^k | k \in \mathbb{Z}\}$: T-invariance of the measure μ ensures that the state defined by μ can be taken as an equilibrium state with respect to the given dynamics.

All phase trajectories passing through $\boldsymbol{x} \in \mathcal{X}$ at time 0 can be encoded into sequences $\{T^k \boldsymbol{x}\}_{k\in\mathbb{Z}}$ [7].

Classical dynamical systems are thus conveniently described by measure-theoretic triplets $(\mathcal{X}, \mu, T)$. In particular, in the present work, we shall focus upon the following choices:

- $\mathcal{X}$: the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \pmod 1\}$;
- μ: the Lebesgue measure, $\mu(\mathrm{d}\boldsymbol{x}) = \mathrm{d}x_1\, \mathrm{d}x_2$, on $\mathbb{T}^2$;
- T: the invertible measurable transformations on $\mathbb{T}^2$ represented by a modular matrix action, as follows:

$$T(\boldsymbol{x}) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \pmod 1, \qquad \begin{array}{l} t_{ij} \in \mathbb{Z}, \quad \forall (i, j) \in \{1, 2\}^2 \\ \det(T) = t_{11}t_{22} - t_{21}t_{12} = 1 \end{array} \tag{1a}$$

$$T^{-1}(\boldsymbol{x}) = \begin{pmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \pmod 1. \tag{1b}$$

Remark 2.1.

(i) In the following, a point $\boldsymbol{x}$ of the torus will correspond to an equivalence class of $\mathbb{R}^2$ points whose coordinates differ by integer values;

(ii) in (1) we use brackets to distinguish between the mere matrix action $T \cdot \boldsymbol{x}$ and the (mod 1) one $T(\boldsymbol{x})$;

(iii) $T = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is known as an Arnold cat map [7], and it is an element of $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{M}_2(\mathbb{Z})$, where the latter is the subset of 2×2 matrices with integer entries, $\mathrm{GL}_2(\mathbb{Z})$ is the subset of invertible matrices and $\mathrm{SL}_2(\mathbb{Z})$ is the subset of matrices with determinant 1;

(iv) the dynamics generated by $T \in \mathrm{SL}_2(\mathbb{Z})$, that is the one we are focusing on, is called *unimodular group* [7] (UMG for short);

(v) since $\det(T) = 1$, the Lebesgue measure μ is *invariant* for all $T^n \in \mathrm{SL}_2(\mathbb{Z})$, $n \in \mathbb{Z}$.

In order to develop an algebraic discretization procedure as in [21], it proves convenient to follow an algebraic approach and replace $(\mathbb{T}^2, \mu, T)$ with the algebraic triple $\left(L^\infty_\mu(\mathbb{T}^2), \omega_\mu, \Theta\right)$, where

$L^\infty_\mu(\mathbb{T}^2)$ is the (Abelian) von Neumann *-algebra of (equivalence classes of) essentially bounded functions on $\mathbb{T}^2$ [22, 23], equipped with the so-called essential supremum norm $\|\cdot\|_\infty$ [24];

ω_μ is the state (expectation) on $L^\infty_\mu(\mathbb{T}^2)$, defined by the reference measure μ as

$$\omega_\mu : L^\infty_\mu(\mathbb{T}^2) \ni f \longmapsto \omega_\mu(f) := \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{x}) f(x) \in \mathbb{R}^+; \tag{2}$$

Θ is the automorphism of $L^\infty_\mu(\mathbb{T}^2)$ defined by $\Theta^j(f) := f \circ T^j$, satisfying $\omega \circ \Theta^j = \omega$.

2.1. Discretization of phase space

From an algebraic point of view, a discretization procedure very much resembles quantization. Given the classical algebraic triple $\left(L^\infty_\mu(\mathbb{T}^2), \omega_\mu, \Theta\right)$, the core of a quantization–dequantization procedure (specifically an $\mathcal{N}$-dimensional quantization) is twofold:

- finding a pair of *-morphisms, $\mathcal{J}_{\mathcal{N},\infty}$ mapping $L^\infty_\mu(\mathbb{T}^2)$ into a finite-dimensional algebra $\mathcal{M}_\mathcal{N}$ (in general a full $N \times N$ matrix algebra) and $\mathcal{J}_{\infty,\mathcal{N}}$ mapping $\mathcal{M}_\mathcal{N}$ backward into $L^\infty_\mu(\mathbb{T}^2)$;
- providing an automorphism $\Theta_\mathcal{N}$, the quantum dynamics, acting on $\mathcal{M}_\mathcal{N}$ such that it approximates in a suitable sense the classical one, Θ, on $L^\infty_\mu(\mathbb{T}^2)$ as follows:

$$\mathcal{J}_{\infty,\mathcal{N}} \circ \Theta^j_\mathcal{N} \circ \mathcal{J}_{\mathcal{N},\infty} \underset{N\to\infty}{\longrightarrow} \Theta^j.$$

The latter requirement can be seen as a modification of the so-called Egorov property (see [25]).

A similar procedure, that we will call *discretization*, can be obtained if we replace the full matrix algebra $\mathcal{M}_\mathcal{N}$ with a finite Abelian one, namely the algebra $\mathcal{D}_\mathcal{N}$ consisting of $N^2 \times N^2$ diagonal matrices.

In order to give to elements of $\mathcal{D}_\mathcal{N}$ the meaning of *discrete observables*, we define a suitable Hilbert space: to do this, we consider a discretized version of $(\mathbb{T}^2, \mu, T)$ which arises by forcing the continuous classical system to live on a square lattice $L_N \subseteq \mathbb{T}^2$ of spacing $\frac{1}{N}$:

$$L_N := \left\{ \frac{\boldsymbol{p}}{N} \middle| \boldsymbol{p} \in (\mathbb{Z}/N\mathbb{Z})^2 \right\}, \tag{3}$$

where $(\mathbb{Z}/N\mathbb{Z})$ denotes the residual class (mod N), that is $0 \leqslant p_i \leqslant N-1$.

Now we take the $\mathcal{N} := N^2$ points of L_N as labels of the elements $\{|\boldsymbol{\ell}\rangle\}_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2}$ of an orthonormal basis (o.n.b.) of the $\mathcal{N}$-dimensional Hilbert space $\mathcal{H}_\mathcal{N}$, and we consider discrete algebraic triples $(\mathcal{D}_\mathcal{N}, \tau_\mathcal{N}, \Theta_\mathcal{N})$, consisting of

$\mathcal{D}_\mathcal{N}$: an $\mathcal{N} \times \mathcal{N}$ matrix algebra diagonal in the orthonormal basis introduced above;

$\tau_\mathcal{N}$: the uniform state (expectation) on $\mathcal{D}_\mathcal{N}$ defined by

$$\tau_\mathcal{N} : \mathcal{D}_\mathcal{N} \ni D \longmapsto \tau_\mathcal{N}(D) := \frac{1}{\mathcal{N}} \mathrm{Tr}(D) \in \mathbb{R}^+; \tag{4}$$

$\Theta_\mathcal{N}$: an automorphism of $\mathcal{D}_\mathcal{N}$ suitably reproducing Θ when $N \longrightarrow \infty$ (see section 2.2).

In particular, as the anti-Wick quantization can be obtained by means of coherent states [12], a similar anti-Wick discretization of $\left(L^\infty_\mu(\mathbb{T}^2), \omega_\mu, \Theta\right)$ in $(\mathcal{D}_\mathcal{N}, \tau_\mathcal{N}, \Theta_\mathcal{N})$ can be performed [21] once we specify what we consider as 'coherent states' on $\mathcal{H}_\mathcal{N}$, and this is the purpose of next section.

Intuitively, a discrete description of $(\mathbb{T}^2, \mu, T)$ becomes finer when we increase N, the number of points per linear dimension on the grid L_N in (3): this corresponds to enlarging the dimension of the Hilbert space $\mathcal{H}_\mathcal{N}$ associated with the corresponding algebraic triple $(\mathcal{D}_\mathcal{N}, \tau_\mathcal{N}, \Theta_\mathcal{N})$. In this sense, the lattice spacing $a := \frac{1}{N}$ of the grid L_N is a natural *discretization parameter* playing an analogous role to the *quantization parameter* $\hbar$.

2.2. Lattice states on $\mathcal{H}_\mathcal{N}$

In analogy with the properties of quantum coherent states, we shall look for analogous states on the torus that we shall call lattice states [21]. For the benefit of the reader, we list below the set of properties which make quantum coherent states such a useful tool in semiclassical analysis.

Properties 2.1 (of quantum coherent states)**.** *A family* $\{|C_\mathcal{N}(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\} \in \mathcal{H}_\mathcal{N}$ *of vectors, indexed by points* $\boldsymbol{x} \in \mathbb{T}^2$*, constitutes a set of coherent states on the torus if it satisfies the following requirements:*

(1) Measurability: $\boldsymbol{x} \mapsto |C_\mathcal{N}(\boldsymbol{x})\rangle$ *is measurable on* $\mathbb{T}^2$*;*
(2) Normalization: $\|C_\mathcal{N}(\boldsymbol{x})\|^2 = 1, \boldsymbol{x} \in \mathbb{T}^2$*;*
(3) Completeness: $\mathcal{N} \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{x}) |C_\mathcal{N}(\boldsymbol{x})\rangle\langle C_\mathcal{N}(\boldsymbol{x})| = \mathbb{1}$*;*
(4) Localization: given $\varepsilon > 0$ *and* $d_0 > 0$*, there exists* $N_0(\varepsilon, d_0)$ *such that for* $N \geqslant N_0(\varepsilon, d_0)$ *and* $d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}) \geqslant d_0$ *one has* $\mathcal{N}|\langle C_\mathcal{N}(\boldsymbol{x}), C_\mathcal{N}(\boldsymbol{y})\rangle|^2 \leqslant \varepsilon$*.*

The symbol $d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y})$ used in the localization property stands for the length of the shorter segment connecting the two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}^2$, namely we shall denote by

$$d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}) := \min_{\boldsymbol{n} \in \mathbb{Z}^2} \|\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{n}\|_{\mathbb{R}^2} \tag{5}$$

the distance on $\mathbb{T}^2$.

Remark 2.2 (topology of the UMG on the torus)**.**

(i) Note that $d_{\mathbb{T}^2}(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|_{\mathbb{R}^2}$ if $\|\boldsymbol{a} - \boldsymbol{b}\|_{\mathbb{R}^2} \leqslant \frac{1}{2}$.
(ii) All the automorphisms $T \in \mathrm{SL}_2(\mathbb{Z})$ defined in (1) act continuously on the torus, when the topology is given by the distance (5).

Resorting to the decomposition $\mathbb{T}^2 \ni \boldsymbol{x} = \left(\frac{\lfloor Nx_1 \rfloor}{N}, \frac{\lfloor Nx_2 \rfloor}{N}\right) + \left(\frac{\langle Nx_1 \rangle}{N}, \frac{\langle Nx_2 \rangle}{N}\right) =: \frac{\lfloor N\boldsymbol{x} \rfloor}{N} + \frac{\langle N\boldsymbol{x} \rangle}{N}$, where $\lfloor \cdot \rfloor$ and $\langle \cdot \rangle$ denote the integer, respectively fractional, part of a real number, we now make use of the definition of the family $|C_\mathcal{N}(\boldsymbol{x})\rangle$ of lattice states given in [21] that consists in associating with points of $\mathbb{T}^2$ specific lattice points (see [21], figure 1).

Definition 2.1 (lattice states)**.** *Given* $\boldsymbol{x} \in \mathbb{T}^2$*, we shall denote by* $\hat{\boldsymbol{x}}_N$ *the element of* $(\mathbb{Z}/N\mathbb{Z})^2$ *given by*

$$\hat{\boldsymbol{x}}_N = (\hat{x}_{N,1}, \hat{x}_{N,2}) := \left(\left\lfloor Nx_1 + \tfrac{1}{2} \right\rfloor, \left\lfloor Nx_2 + \tfrac{1}{2} \right\rfloor\right), \tag{6}$$

and call lattice states on $\mathbb{T}^2$ *the vectors* $|C_\mathcal{N}(\boldsymbol{x})\rangle$ *defined by*

$$\mathbb{T}^2 \ni \boldsymbol{x} \mapsto |C_\mathcal{N}(\boldsymbol{x})\rangle := |\hat{\boldsymbol{x}}_N\rangle \in \mathcal{H}_\mathcal{N}. \tag{7}$$

The reader can check in [21] that family $\{|C_{\mathcal{N}}(\boldsymbol{x})\}$ satisfies properties 1.1. In particular, in the last proof, it is also shown that, due to our particular choice of lattice states, we have a stronger localization than in property 2.1 (4), namely

(4′). *Localization: given* $d_0 > 0$, *there exists* $N_0(d_0)$ *such that for* $N \geqslant N_0(d_0)$ *and* $d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}) \geqslant d_0$ *one has* $\langle C_{\mathcal{N}}(\boldsymbol{x}), C_{\mathcal{N}}(\boldsymbol{y})\rangle = 0$.

2.3. Anti-Wick discretization and its continuous limit on $\mathbb{T}^2$

In order to study the continuous limit and, more generally, the quasi-continuous behaviour of $(\mathcal{D}_{\mathcal{N}}, \tau_{\mathcal{N}}, \Theta_{\mathcal{N}})$ when $N \to \infty$, we follow the semiclassical technique known as anti-Wick quantization. Therefore, we start by choosing concrete discretization/dediscretization *-morphisms.

Definition 2.2. *Given the family of lattice states* $\{|C_{\mathcal{N}}(\boldsymbol{x})\rangle\} \in \mathcal{H}_{\mathcal{N}}$ *of the previous section, the anti-Wick-like discretization scheme (*AW*, for short) is described by a one-parameter family of (completely) positive unital map* $\mathcal{J}_{\mathcal{N},\infty} : L^\infty_\mu(\mathbb{T}^2) \to \mathcal{D}_{\mathcal{N}}$

$$L^\infty_\mu(\mathbb{T}^2) \ni f \mapsto \mathcal{N} \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{x}) f(\boldsymbol{x}) |C_{\mathcal{N}}(\boldsymbol{x})\rangle\langle C_{\mathcal{N}}(\boldsymbol{x})| =: \mathcal{J}_{\mathcal{N},\infty}(f) \in \mathcal{D}_{\mathcal{N}}.$$

The corresponding dediscretization operation is described by the (completely) positive unital map $\mathcal{J}_{\infty,\mathcal{N}} : \mathcal{D}_{\mathcal{N}} \to L^\infty_\mu(\mathbb{T}^2)$

$$\mathcal{D}_{\mathcal{N}} \ni X \mapsto \langle C_{\mathcal{N}}(\boldsymbol{x}), X C_{\mathcal{N}}(\boldsymbol{x})\rangle =: \mathcal{J}_{\infty,\mathcal{N}}(X)(\boldsymbol{x}) \in L^\infty_\mu(\mathbb{T}^2).$$

Both maps are identity preserving (unital) because of the conditions satisfied by the family of lattice states and completely positive too, since both $L^\infty_\mu(\mathbb{T}^2)$ and $\mathcal{D}_{\mathcal{N}}$ are commutative algebras. The reader can find in [14, 21] a list of simple properties of these maps that incorporate minimal requests for rigorously defining the sense in which the discrete dynamical systems $(\mathcal{D}_{\mathcal{N}}, \tau_{\mathcal{N}}, \Theta_{\mathcal{N}})$ tends to $\left(L^\infty_\mu(\mathbb{T}^2), \omega_\mu, \Theta\right)$, when $\frac{1}{N} \to 0$.

3. Discretization of the dynamics

3.1. General properties of matrix actions on the plane

The next natural step in our discretization procedure will be the definition of a suitable discrete dynamics $\Theta_{\mathcal{N}}$ on the Abelian algebra $\mathcal{D}_{\mathcal{N}}$ of section 1.1. Before doing this we shall focus on some basic properties of the (integer) matrix action on the plane that are

$$\mathbb{R}^2 \ni \boldsymbol{x} \longmapsto T\boldsymbol{x} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \qquad \begin{array}{l} t_{\imath\jmath} \in \mathbb{Z}, \quad \forall\, (\imath, \jmath) \in \{1, 2\}^2 \\ \det(T) = t_{11}t_{22} - t_{21}t_{12} = 1. \end{array}$$

Note that in this section we begin by considering integer matrices T, with determinant 1, mapping the plane onto itself; in section 2.2, we will go back to actions on the torus $\mathbb{T}^2$, as in (1a).

Definitions 3.1 (families of matrix actions)**.** *We exclude from now on the cases* $T = \pm\mathbb{1}_2$, *the identity on the plane, that are trivial. Depending on the trace of T we have three families of maps, characterized by their spectral properties; in particular, denoting with* $t := \frac{\mathrm{Tr}(T)}{2}$ *the semi-trace of T, the eigenvalues are given by* $t \pm \sqrt{t^2 - 1}$ *and we have*

$|t| > 1$ *(hyperbolic family). One eigenvalue of T,* λ, *is greater than 1 (in modulus) and the other one is* λ^{-1}. *In this case, distances are stretched along the direction of the eigenvector* $|\boldsymbol{e}_+\rangle$, $T|\boldsymbol{e}_+\rangle = \lambda|\boldsymbol{e}_+\rangle$, *contracted along that of* $|\boldsymbol{e}_-\rangle$, $T|\boldsymbol{e}_-\rangle = \lambda^{-1}|\boldsymbol{e}_-\rangle$. *The (positive) Lyapunov exponent is given by* $\xi = \log|\lambda|$.

$|\boldsymbol{t}| = \mathbf{1}$ *(parabolic family). There is only one eigenvalue, whose modulus is equal to 1, which corresponds to an eigenvector* $|\boldsymbol{e}_0\rangle$.

$|\boldsymbol{t}| < \mathbf{1}$ *(elliptic family). The two eigenvalues are conjugate complex numbers* $\mathrm{e}^{\mathrm{i}\phi}$ *and* $\mathrm{e}^{-\mathrm{i}\phi}$*, whose corresponding eigenvectors* $|\boldsymbol{e}_+\rangle$ *and* $|\boldsymbol{e}_-\rangle$ *are complex conjugate vectors of* $\mathbb{C}^2$*. On the (non-orthogonal) basis* $\{|\boldsymbol{e}_\mathrm{R}\rangle, |\boldsymbol{e}_\mathrm{I}\rangle\} := \{\mathrm{Re}(|\boldsymbol{e}_+\rangle), \mathrm{Im}(|\boldsymbol{e}_+\rangle)\}$, T^n *is represented by means of the rotation matrix*

$$R^n = \begin{pmatrix} \cos(n\phi) & \sin(n\phi) \\ -\sin(n\phi) & \cos(n\phi) \end{pmatrix}. \tag{8}$$

Before exploring the properties of the three regimes given above, we now list some more

Definition 3.2. *Let* $B_T(0) := \{\boldsymbol{x} \in \mathbb{R}^2 | \|\boldsymbol{x}\|_{\mathbb{R}^2} \leqslant 1\}$ *be the unitary ball on the plane and*

$$B_T(p) := \{\boldsymbol{x} \in \mathbb{R}^2 \mid T^{-p}\boldsymbol{x} \in B_T(0)\} \tag{9}$$

be the p-evolved ball $(p \in \mathbb{Z})$*. Then define as*

$$B_T^{(n)} := \bigcup_{p=-n}^{n} B_T(p) \tag{10}$$

the union of all evolved balls from time $-n$ *up to time* n $(n \in \mathbb{N})$ *and let* $D_T^{(n)} := \mathrm{diam}\big[B_T^{(n)}\big]$ *be its diameter, so that* $D_T(p) := \mathrm{diam}[B_T(p)]$ *will be the diameter of the p-evolved ball* $(\mathrm{diam}[E] := \sup_{\boldsymbol{x},\boldsymbol{y}\in E} \|\boldsymbol{x}-\boldsymbol{y}\|_{\mathbb{R}^2})$*. Further, we denote by* η *the largest eigenvalue of the matrix* $|T| = \sqrt{T^\dagger T}$.

Using this notation we now list three propositions, one for each family, that incorporate the main properties; a sketch of their proofs is given in appendix A.

Proposition 3.1 (hyperbolic family)**.** *Let T be a matrix belonging to the hyperbolic family of definitions 3.1.*

Without loss of generality, we choose $|\boldsymbol{e}_+\rangle$ *and* $|\boldsymbol{e}_-\rangle$ *in such a way that the angle* β *from the former to the latter lies in* $(0, \pi)$ *and we fix an orthogonal reference system* $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ *with x-axis oriented along the eigenvector* $|\boldsymbol{e}_+\rangle$*: in such a system all orbits of the (discrete) group* $\{T^k\}_{k\in\mathbb{Z}}$ *lie on hyperbolas*

$$y^2 \cos\beta - xy \sin\beta = \mathrm{const}. \tag{11}$$

The angle β*, whose sine is positive according to our choice of* $|\boldsymbol{e}_+\rangle$ *and* $|\boldsymbol{e}_-\rangle$*, is related to* η *of definitions 3.2 by*

$$\sin\beta = \frac{\lambda - \lambda^{-1}}{\eta - \eta^{-1}}; \tag{12}$$

moreover, for every $n \in \mathbb{N}$*, the set* $B_T^{(n)}$ *is confined into the hyperbolic region delimited by the four branches of the two hyperbolas*

$$2y^2 \cos\beta - 2xy \sin\beta - (\cos\beta \pm 1) = 0. \tag{13}$$

For the diameters, we have

$$D_T^{(n)} = D_T(n) = \frac{\lambda^n - \lambda^{-n}}{2\sin\beta} \left\{ 1 + \sqrt{1 + \left(\frac{2\sin\beta}{\lambda^n - \lambda^{-n}} \right)^2} \right\} \tag{14}$$

or, resorting to the expression for the Lyapunov exponent ξ *given in definition 3.1:*

$$\sin\beta \sinh\big\{\log\big[D_T^{(n)}\big]\big\} = \sinh(n\xi). \tag{15}$$

Moreover,

$$\forall n \in \mathbb{N}, \qquad D_T^{(n)} \leqslant \frac{\lambda^n}{\sin\beta} \qquad \text{and} \qquad D_T^{(n)} \xrightarrow[n\longrightarrow\infty]{} \frac{\lambda^n}{\sin\beta}. \tag{16}$$

Proposition 3.2 (parabolic family)**.** *Let T be a matrix belonging to the parabolic family of definitions 3.1.*

We fix an orthogonal reference system $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ with x-axis oriented along the eigenvector $|\boldsymbol{e}_0\rangle$: in such a system all orbits of the (discrete) group $\{T^k\}_{k\in\mathbb{Z}}$ lie on the

$$\begin{cases} \text{line} & y = \text{const} \quad \text{if} \quad t = +1 \\ \text{two lines} & y^2 = \text{const} \quad \text{if} \quad t = -1. \end{cases} \tag{17}$$

For every $n \in \mathbb{N}$ the set $B_T^{(n)}$ is confined into the stripe delimited by the two lines

$$y^2 = 1. \tag{18}$$

Resorting to η of definitions 3.2, we introduce a positive real parameter

$$J = \frac{\eta - \eta^{-1}}{2} \tag{19}$$

that is used in the expression for the diameters, that is

$$D_T^{(n)} = D_T(n) = nJ + \sqrt{n^2J^2 + 1} \tag{20}$$

or, equivalently,

$$\sinh\left\{\log\left[D_T^{(n)}\right]\right\} = nJ. \tag{21}$$

Moreover,

$$\forall n \in \mathbb{N}, \qquad D_T^{(n)} \leqslant 2nJ + 1 \tag{22}$$

and

$$D_T^{(n)} \xrightarrow[n\longrightarrow\infty]{} 2nJ. \tag{23}$$

Proposition 3.3 (elliptic family)**.** *Let T be a matrix belonging to the elliptic family of definitions 3.1; if the entries of this matrix are integer, it holds true:*

$$\forall n \in \mathbb{N}, \qquad D_T(n) \leqslant \eta, \tag{24}$$

$$\forall n \in \mathbb{N}^+, \qquad D_T^{(n)} = \eta, \tag{25}$$

where η is the one introduced in definitions 3.2.

3.2. Algebraic description of discretized UMG

Our aim is now to define a suitable discrete evolution $\Theta_{\mathcal{N}}$ on $\mathcal{D}_{\mathcal{N}}$ (see section 2.1 for the definitions), such that the discretized triplets $(\mathcal{D}_{\mathcal{N}}, \tau_{\mathcal{N}}, \Theta_{\mathcal{N}})$ converge to the continuous one $\left(L^\infty_\mu(\mathbb{T}^2), \omega_\mu, \Theta\right)$.

We start by introducing a new family of maps $\left\{U_T^j\right\}_{j\in\mathbb{Z}}$, defined on the torus $\mathbb{T}^2([0, N))$, given by the action determined by the matrix $T \pmod N$, that is

$$\mathbb{T}^2([0, N)) \ni \boldsymbol{x} \longmapsto U_T^j(\boldsymbol{x}) := NT^j\left(\frac{\boldsymbol{x}}{N}\right) \in \mathbb{T}^2([0, N)), \qquad j \in \mathbb{Z}, \tag{26}$$

where $T(\cdot)$ is the map defined in (1). The $U_T^j(\cdot)$ maps are extensions of the $T^j(\cdot)$ maps on the enlarged torus $\mathbb{T}^2([0, N))$; moreover, they do map the lattice $(\mathbb{Z}/N\mathbb{Z})^2$ into itself, so that the maps $T^j(\cdot)$ do it with the lattice L_N of (3).

Note that the map $(\mathbb{Z}/N\mathbb{Z})^2 \ni \boldsymbol{\ell} \longmapsto U_T(\boldsymbol{\ell}) \in (\mathbb{Z}/N\mathbb{Z})^2$ is a bijection.

Definition 3.3. $\Theta_{\mathcal{N}}$ *will denote the map*

$$\mathcal{D}_{\mathcal{N}} \ni X \longmapsto \Theta_{\mathcal{N}}(X) := \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{U_T(\boldsymbol{\ell}),U_T(\boldsymbol{\ell})} |\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| \in \mathcal{D}_{\mathcal{N}}.$$

The map $\Theta_{\mathcal{N}}$ is a *-automorphism of $\mathcal{D}_{\mathcal{N}}$; indeed

$$\begin{aligned}\Theta_{\mathcal{N}}(X) &= \sum_{U_T^{-1}(\boldsymbol{s})\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{s},\boldsymbol{s}} \big|U_T^{-1}(\boldsymbol{s})\big\rangle\big\langle U_T^{-1}(\boldsymbol{s})\big| \\ &= W_{T,N}\left(\sum_{\substack{\text{all equiv.}\\ \text{classes}}} X_{\boldsymbol{s},\boldsymbol{s}}|\boldsymbol{s}\rangle\langle\boldsymbol{s}|\right)W_{T,N}^* \\ &= W_{T,N}\, X\, W_{T,N}^*,\end{aligned}$$

where the operators $W_{T,N}$, defined by linearly extending the maps

$$\mathcal{H}_{\mathcal{N}} \ni |\boldsymbol{\ell}\rangle \longmapsto W_{T,N}|\boldsymbol{\ell}\rangle := \big|U_T^{-1}(\boldsymbol{\ell})\big\rangle \in \mathcal{H}_{\mathcal{N}} \tag{27}$$

to $\mathcal{H}_{\mathcal{N}}$, are unitary: $W_{T,N}^*|\boldsymbol{\ell}\rangle := |U_T(\boldsymbol{\ell})\rangle$. For the same reason, $\tau_{\mathcal{N}}$ is a $\Theta_{\mathcal{N}}$-invariant state.

4. Continuous limit of the dynamics

One of the main issues in the semiclassical analysis is to compare if and how the quantum and classical time evolutions mimic each other when the quantization parameter goes to zero.

In this paper, we are instead considering the possible agreement between the dynamics of continuous classical systems and that of a class of discrete approximants. In practice, in our case, we will study the difference

$$\Theta^j - \mathcal{J}_{\infty,\mathcal{N}} \circ \Theta_{\mathcal{N}}^j \circ \mathcal{J}_{\mathcal{N},\infty} \tag{28}$$

which represents how much the discrete dynamics at time step j differs from the continuous one at the same time step.

For quantum systems, whose classical limit is chaotic, the situation is strikingly different from those with regular classical limit. In the former case, classical and quantum mechanics agree, that is a difference as in (28) is negligible, only over times j which scale logarithmically (and not as a power law) in the quantization parameter.

As we shall see, such kind of scaling is not exclusively related to non-commutativity; in fact, the quantization-like procedure developed so far exhibits similar behaviour when $N \to \infty$ and we recover $\left(L_\mu^\infty(\mathbb{T}^2), \omega_\mu, \Theta\right)$ as a continuous limit of $(\mathcal{D}_{\mathcal{N}}, \tau_{\mathcal{N}}, \Theta_{\mathcal{N}})$.

4.1. Continuous limit of discretized UMG

We want to show that the difference in (28) goes to zero in a suitable topology, at least on a certain time scale. Such scales, commonly called *breaking times*, depend on the family of the considered map T. In the following, we give three different scaling functions of n, one per each family of matrix action, that will be compared with $\log N$ in the joint limits in n and N that we will construct in this section.

Definition 4.1. *We shall denote by* $\Gamma_T(n)$ *the* scaling function *of time associated with a map* T. *In particular, in the different families of definition 3.1, it is given by*

$$\Gamma_T(n) = \begin{cases} \log(\lambda^n) & \text{for the } \textit{hyperbolic family} \text{ of } T \\ \log n & \text{for the } \textit{parabolic family} \text{ of } T \\ 0 & \text{for the } \textit{elliptic family} \text{ of } T. \end{cases}$$

We shall concretely show that the difference (28) goes to zero with $N \to \infty$ in the strong topology over the Hilbert space $L^2_\mu(\mathbb{T}^2)$. More precisely, we have

Theorem 1. *Let* $(\mathcal{D}_\mathcal{N}, \tau_\mathcal{N}, \Theta_\mathcal{N})$ *be a sequence of discretized dynamical systems as defined in section 3: for all* $\gamma > 1$,

$$\forall f \in L^\infty_\mu(\mathbb{T}^2), \qquad \operatorname*{s\text{-}lim}_{\substack{j,N\to\infty \\ \Gamma_T(j)<\frac{\log N}{\gamma}}} \left(\Theta^j - \mathcal{J}_{\infty,\mathcal{N}} \circ \Theta^j_\mathcal{N} \circ \mathcal{J}_{\mathcal{N},\infty}\right)(f) = 0, \tag{29}$$

where the limit is in the strong topology over the Hilbert space $L^2_\mu(\mathbb{T}^2)$.

The previous theorem indicates that the time limit and the continuous limit do not commute in the parabolic and hyperbolic cases. In particular, the difference between the discretized dynamics and the continuous one can be made small by increasing N, while it becomes large beyond the time scale $\Gamma_T(j) \simeq \log N$. This phenomenon is the same as in quantum chaos and points to discretization of phase space (in the traditional semiclassical treatment of quantum systems), rather than to non-commutativity, as the source of the so-called *logarithmic breaking time* for hyperbolic systems. The constant γ is a *form factor*, which reflects the fine structure of the dynamics: for instance, in the case of quantum cat maps [14], $\gamma = 2$.

For the elliptic case $\operatorname*{s\text{-}lim}_{\substack{j,N\to\infty \\ \Gamma_T(j)<\frac{\log N}{\gamma}}} = \operatorname*{s\text{-}lim}_{\substack{j,N\to\infty \\ 0<\frac{\log N}{\gamma}}}$ means $\operatorname{s\text{-}lim}_{j,N\to\infty}$; $0 < \log N$ is just a way to write that we do not consider any relation between j and N. We adopted this in order to have uniformity among the notation in the three different families of matrix action.

The constraint $j \leqslant C \log \mathcal{N}$ is typical of *hyperbolic* behaviour with Lyapunov exponent $\log \lambda$ and comes heuristically as follows: the expansion of an initial small distance δ can be exponential until the distance becomes the largest possible, namely $\delta\lambda^{T_\mathrm{B}} \simeq 1$ (on the torus). After discretization, the minimal distance gives $\delta = \frac{1}{N}$, therefore one estimates $T_\mathrm{B} \simeq \frac{\log N}{\log \lambda}$, which is called *breaking time* and sets the time scale over which continuous and discretized dynamics mimic each other.

In quantum chaos, the semiclassical analysis leads to an estimate of T_B exactly as above; further, the logarithmic dependence on $\hbar$ of T_B is a signature of the hyperbolic character of the classical limit. Conversely, if the classical limit is regular (parabolic and elliptic case), then the time scale when quantum and classical behaviour is more or less indistinguishable goes in general as $\hbar^{-b}$, $b > 0$.

The proof of theorem 1 consists of several steps, among which the most important is a property, satisfied by our choice of lattice states, which we shall call dynamical localization. We give a full proof that the lattice states satisfy such property, since it represents a natural request that should be fulfilled by any consistent discretization/dediscretization (quantization/dequantization) scheme; before giving the statement of the dynamical localization condition, let us introduce one more

Definition 4.2. *We shall denote by* $K_{N,n}(\boldsymbol{x}, \boldsymbol{y})$ *the quantity*

$$K_{N,n}(\boldsymbol{x}, \boldsymbol{y}) := \langle C_\mathcal{N}(\boldsymbol{x})\, W^n_{T,N} C_\mathcal{N}(\boldsymbol{y})\rangle = \langle U^n_T(\hat{\boldsymbol{x}}_N), \hat{\boldsymbol{y}}_N\rangle,$$

where $W^{j}_{T,N}$ is the unitary operator defined in (27) and $\{|C_{\mathcal{N}}(\boldsymbol{x})\rangle\}$ is the set of LS *of definition 2.1.*

Theorem 2 (dynamical localization with $\{|C_{\mathcal{N}}(\boldsymbol{x})\rangle\}$ states)**.** *For every $\gamma > 1$ and $d_0 > 0$, there exists $N_0 = N_0(\gamma, d_0) \in \mathbb{N}^+$ with the following property: if $N > N_0$ and $\Gamma_T(n) < \frac{\log N}{\gamma}$, then*

$$d_{\mathbb{T}^2}(T^n(\boldsymbol{x}), \boldsymbol{y}) \geqslant d_0 \Longrightarrow K_{N,n}(\boldsymbol{x}, \boldsymbol{y}) = 0,$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}^2$, where $K_{N,n}(\boldsymbol{x}, \boldsymbol{y})$ are those of definition 4.2 and the scaling function of time $\Gamma_T(n)$ has been introduced in definition 4.1.

In analogy to the quantum case, dynamical localization is what one expects from a good choice of states suited to the study of the continuous limit: in fact, it essentially amounts to asking that LS remain decently localized around the continuous trajectories while evolving with the corresponding discrete evolution. As we shall see, this is the case only in time such that $\Gamma_T(n) < (\log N)/\gamma$. Informally, when $N \to \infty$, the quantities $K_{N,j}(\boldsymbol{x}, \boldsymbol{y})$ should behave as if $\mathcal{N}|K_{N,j}(\boldsymbol{x}, \boldsymbol{y})|^2 \simeq \delta(T^j(\boldsymbol{x}) - \boldsymbol{y})$ and this is the content of the, next proposition 4.1 which will be used in section 5.4.

This would make the discretization analogous to the notion of *regular quantization* described in section V of [19]. Actually, with our choice of LS, the quantity $K_{N,j}(\boldsymbol{x}, \boldsymbol{y})$ is a Kronecker delta.

Proposition 4.1. *Using the same notation as for theorem 2 we have that, for any given real number $\gamma > 1$ and $f \in L^\infty_\mu(\mathbb{T}^2)$, it holds true:*

$$\lim_{\substack{n,N\to\infty \\ \Gamma_T(n)<\frac{\log N}{\gamma}}} \left\| \mathcal{N} \int_{\mathbb{T}^2} f(\boldsymbol{y}) |K_{N,n}(\cdot, \boldsymbol{y})|^2 \, \mu(\mathrm{d}\boldsymbol{y}) - f(T^n(\cdot)) \right\|_2 = 0,$$

where $\|\cdot\|_2$ denotes the $L^2_\mu(\mathbb{T}^2)$-norm.

Proof. The equation of the statement can be expressed in terms of the discretization–dediscretization operators $\mathcal{J}_{\mathcal{N},\infty}$ and $\mathcal{J}_{\infty,\mathcal{N}}$ of definition 2.2, the discrete evolution automorphism $\Theta_{\mathcal{N}}$ of definition 3.3 and the continuous one Θ of section 2 as follows:

$$\lim_{\substack{n,N\to\infty \\ \Gamma_T(n)<\frac{\log N}{\gamma}}} \left\| \left(\Theta^n - \mathcal{J}_{\infty,\mathcal{N}} \circ \Theta^n_{\mathcal{N}} \circ \mathcal{J}_{\mathcal{N},\infty}\right)(f) \right\|_2 = 0.$$

The last equation is proved in the proof of theorem 1 (see (44)). □

In order to prove theorem 2, we need the following auxiliary result.

Proposition 4.2. *Resorting to the distance (5), $\hat{\boldsymbol{x}}_N$ of definition 1.1, U_T of (26) and $(\lambda, \beta, J, \eta)$ used in propositions 3.1–3.3, the following three statements hold:*

For $\boldsymbol{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}^+$

(1) if T is hyperbolic and $N > \widetilde{N}_{\mathrm{hyp}}(n) := \sqrt{2}\frac{\lambda^n}{\sin\beta}$

$$\text{then} \qquad d_{\mathbb{T}^2}\left(T^p(\boldsymbol{x}), \frac{U^p_T(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{\widetilde{N}_{\mathrm{hyp}}(n)}{2N}, \qquad \forall p \leqslant n; \tag{30}$$

(2) if T is parabolic and $N > \widetilde{N}_{\mathrm{par}}(n) := \sqrt{2}\,(2nJ + 1)$

$$\text{then} \qquad d_{\mathbb{T}^2}\left(T^p(\boldsymbol{x}), \frac{U^p_T(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{\widetilde{N}_{\mathrm{par}}(n)}{2N}, \qquad \forall p \leqslant n; \tag{31}$$

(3) if T is elliptic and $N > \widetilde{N}_{\mathrm{ell}} := \sqrt{2}\eta$

$$\textit{then} \qquad d_{\mathbb{T}^2}\left(T^p(\boldsymbol{x}), \frac{U_T^p(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{\widetilde{N}_{\mathrm{ell}}}{2N}, \qquad \forall p \leqslant n. \tag{32}$$

Proof. For every real number t, we have $0 \leqslant \langle Nt + 1/2\rangle = Nt + 1/2 - \lfloor Nt + 1/2\rfloor < 1$, so that $\left|t - \frac{\lfloor Nt+1/2\rfloor}{N}\right| \leqslant \frac{1}{2N}, \forall t \in \mathbb{R}$. From (6) in definition 2.1, we derive

$$d_{\mathbb{T}^2}\left(\boldsymbol{x}, \frac{\hat{\boldsymbol{x}}_N}{N}\right) \leqslant \frac{1}{\sqrt{2}N}, \qquad \forall \boldsymbol{x} \in \mathbb{T}^2. \tag{33}$$

Let us start by proving the first statement, the other being very similar to it. Using the definition of U_T given in (26), we write

$$\left\|T^p(\boldsymbol{x}) - \frac{U_T^p(\hat{\boldsymbol{x}}_N)}{N}\right\|_{\mathbb{R}^2} = \left\|T^p(\boldsymbol{x}) - T^p\left(\frac{\hat{\boldsymbol{x}}_N}{N}\right)\right\|_{\mathbb{R}^2} = \left\|T^p\left(\boldsymbol{x} - \frac{\hat{\boldsymbol{x}}_N}{N}\right)\right\|_{\mathbb{R}^2}, \tag{34}$$

where in the latter equality we applied the linearity of $T(\cdot)$. As (16) was the maximum allowed spreading for the unit ball $B_T(0)$ under the action of n power of the matrix T, now we have

$$\left\|T^p\left(\boldsymbol{x} - \frac{\hat{\boldsymbol{x}}_N}{N}\right)\right\|_{\mathbb{R}^2} \leqslant \frac{\lambda^p}{\sin\beta}\left\|\boldsymbol{x} - \frac{\hat{\boldsymbol{x}}_N}{N}\right\|_{\mathbb{R}^2} \leqslant \frac{1}{\sqrt{2}N}\frac{\lambda^n}{\sin\beta}, \tag{35}$$

indeed $p \leqslant n$ and we applied (33) together with remark 2.2 (i). In order to replace the first norm in (34) with the toral distance, we apply once more the same remark 2.2 (i), providing that $\frac{1}{\sqrt{2}N}\frac{\lambda^n}{\sin\beta} \leqslant \frac{1}{2}$, that is $N \geqslant N_{\mathrm{hyp}}(n)$.

The other statements (31)–(32) are proved in the same way, substituting in (35) the right expression for the diameters, given for parabolic and elliptic cases from (22), respectively (24). □

Proof of theorem 2. Using the definition of $\{|C_{\mathcal{N}}(\boldsymbol{x})\rangle\}$ in (7), we easily compute

$$\left\langle C_{\mathcal{N}}(\boldsymbol{x}) \,\middle|\, W_{T,N}^n C_{\mathcal{N}}(\boldsymbol{y})\right\rangle = \left\langle \hat{\boldsymbol{x}}_N \,\middle|\, U_T^{-n}(\hat{\boldsymbol{y}}_N)\right\rangle = \delta^{(N)}_{U_T^n(\hat{\boldsymbol{x}}_N), \hat{\boldsymbol{y}}_N}. \tag{36}$$

Using the triangular inequality, we get

$$d_{\mathbb{T}^2}\left(\frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}, \frac{\hat{\boldsymbol{y}}_N}{N}\right) \geqslant d_{\mathbb{T}^2}(T^n(\boldsymbol{x}), \boldsymbol{y}) - d_{\mathbb{T}^2}\left(T^n(\boldsymbol{x}), \frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}\right) - d_{\mathbb{T}^2}\left(\frac{\hat{\boldsymbol{y}}_N}{N}, \boldsymbol{y}\right). \tag{37}$$

Now we split the proof and we begin by focusing on the

Hyperbolic case. Since $d_{\mathbb{T}^2}(T^n(\boldsymbol{x}), \boldsymbol{y}) \geqslant d_0$ by hypothesis, using (33) of the proof of proposition 4.2 and (30), that is

$$N > \widetilde{N}_{\mathrm{hyp}}(n) \implies d_{\mathbb{T}^2}\left(T^n(\boldsymbol{x}), \frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{1}{\sqrt{2}N}\frac{\lambda^n}{\sin\beta}, \tag{38}$$

we can derive from (37) that $d_{\mathbb{T}^2}\left(\frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}, \frac{\hat{\boldsymbol{y}}_N}{N}\right) \geqslant d_0 - \frac{1}{\sqrt{2}N}\frac{\lambda^n}{\sin\beta} - \frac{1}{\sqrt{2}N}$.

The rhs of the previous inequality can always be made strictly greater than zero,

$$d_{\mathbb{T}^2}\left(\frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}, \frac{\hat{\boldsymbol{y}}_N}{N}\right) > 0, \tag{39}$$

by choosing an N greater than

$$N_{\mathsf{M}}(n) = \max\left\{\frac{1}{d_0\sqrt{2}}\left(1 + \frac{\lambda^n}{\sin\beta}\right), \widetilde{N}_{\mathrm{hyp}}(n) = \sqrt{2}\frac{\lambda^n}{\sin\beta}\right\}, \tag{40}$$

so that the condition on the lhs of (38) is also satisfied. From (36) and (39), we have

$$N > N_{\mathsf{M}}(n) \implies \left\langle C_{\mathcal{N}}(\boldsymbol{x}) \,\middle|\, W_{T,N}^n C_{\mathcal{N}}(\boldsymbol{y})\right\rangle = 0. \tag{41}$$

Indeed, if the toral distance between two grid points $(\hat{\boldsymbol{z}}_N, \hat{\boldsymbol{w}}_N)$ is different from zero, they cannot be equal (mod N) and so the periodic Kronecker delta in (36) vanishes.

Since the (non-decreasing) function $N_{\mathsf{M}}(n)$ in (40) is eventually bounded by $\lambda^{\gamma n}$ (γ being strictly greater than one), we define $\overline{n}$ as the time when $N_{\mathsf{M}}(\overline{n}) = \lambda^{\gamma\overline{n}} =: N_0$, and choose $N > N_0$. Thus, if $0 < n < \overline{n}$, then $N > N_0 = N_{\mathsf{M}}(\overline{n}) > N_{\mathsf{M}}(n)$, whereas if $\overline{n} \leqslant n < \frac{1}{\gamma}\frac{\log N}{\log\lambda}$, then $N > \lambda^{\gamma n} > N_{\mathsf{M}}(n)$ and (41) holds for all $0 < n < \frac{1}{\gamma}\frac{\log N}{\log\lambda}$, that is $\Gamma_T(n) < \frac{\log N}{\gamma}$ as in the statement.

Parabolic case. Using now (31), that is

$$N > \widetilde{N}_{\mathrm{par}}(n) \quad\Longrightarrow\quad d_{\mathbb{T}^2}\left(T^n(\boldsymbol{x}), \frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{1}{\sqrt{2}N}(2nJ+1), \tag{42}$$

we obtain from (37) that $d_{\mathbb{T}^2}\big(\frac{U_T^n(\hat{\boldsymbol{x}}_N)}{N}, \frac{\hat{\boldsymbol{y}}_N}{N}\big) \geqslant d_0 - \frac{1}{\sqrt{2}N}(2nJ+1) - \frac{1}{\sqrt{2}N}$.

The rhs of the previous inequality can be made strictly greater than zero, by choosing an N greater than

$$N_{\mathsf{M}}(n) = \max\left\{\frac{\sqrt{2}}{d_0}(nJ+1), \widetilde{N}_{\mathrm{par}}(n) = \sqrt{2}(2nJ+1)\right\}, \tag{43}$$

so that the condition on the lhs of (42) is also satisfied. Reasoning as for the hyperbolic case, we conclude that (41) still holds true in this case and we choose n^γ as the bounding function of the (non-decreasing) $N_{\mathsf{M}}(n)$ of (43).

Finally, as for the hyperbolic case, we define $\overline{n}$ as the time when $N_{\mathsf{M}}(\overline{n}) = \overline{n}^\gamma =: N_0$, and choose $N > N_0$. Thus, if $0 < n < \overline{n}$, then $N > N_0 = N_{\mathsf{M}}(\overline{n}) > N_{\mathsf{M}}(n)$, whereas if $\overline{n} \leqslant n < N^{\frac{1}{\gamma}}$, then $N > n^\gamma > N_{\mathsf{M}}(n)$ and (41) holds for all $0 < n < N^{\frac{1}{\gamma}}$, that is $\Gamma_T(n) < \frac{\log N}{\gamma}$ as in the statement.

Elliptic case. The same strategy adopted in the previous two cases now leads us to define a new N_{M}, *independent* of n, given by $N_{\mathsf{M}} = \max\big\{\frac{1}{d_0\sqrt{2}}(\eta+1), \widetilde{N}_{\mathrm{ell}}(n) = \eta\sqrt{2}\big\}$; thus, for $N > N_{\mathsf{M}}$, the periodic Kronecker delta in (36) vanishes.

The absence of a relation between N and n, for $N > N_{\mathsf{M}}$, is expressed in the relation $\Gamma_T = 0 < \frac{\log N}{\gamma}$, always true for all N. □

We are finally in a position to conclude with

Proof of theorem 1. We will concentrate on the case of continuous f, that is $f \in \mathcal{C}^0(\mathbb{T}^2)$ $\big(\subset L^2_\mu(\mathbb{T}^2)\big)$; the extension to essentially bounded f is straightforward and can be realized by applying Lusin's theorem [23, 24, 26], as the reader can see in [21].

Let $f \in \mathcal{C}^0(\mathbb{T}^2)$ and $\mathrm{Op}_{j,N}(f) := \big(\Theta^j - \mathcal{J}_{\infty,\mathcal{N}} \circ \Theta^j_{\mathcal{N}} \circ \mathcal{J}_{\mathcal{N},\infty}\big)(f)$: note that $\mathrm{Op}_{j,N}(f)$ is a multiplication operator on $L^2_\mu(\mathbb{T}^2)$, but also an $L^\infty_\mu(\mathbb{T}^2)$ (and thus also an $L^2_\mu(\mathbb{T}^2)$) function. According to (29), we must show that

$$\forall g \in L^2_\mu(\mathbb{T}^2), \qquad \lim_{\substack{j,N\to\infty \\ \Gamma_T(j) < \frac{\log N}{\gamma}}} \|\mathrm{Op}_{j,N}(f) g\|_2 = 0.$$

Using Schwartz's inequality first with g in the class of *simple functions* and then using their density in $L^2_\mu(\mathbb{T}^2)$, we have just to show that

$$\lim_{\substack{j,N\to\infty \\ \Gamma_T(j) < \frac{\log N}{\gamma}}} \|\mathrm{Op}_{j,N}(f)\|_2 = 0. \tag{44}$$

In [21] it is shown that

$$\|\mathrm{Op}_{j,N}(f)\|_2^2 = \omega_\mu(|f|)^2 + \tau_{\mathcal{N}}[\mathcal{J}_{\mathcal{N},\infty}(f)^* \mathcal{J}_{\mathcal{N},\infty}(f)] - 2\,\mathrm{Re}(I_{j,N}(f)),$$

with

$$I_{j,N}(f) := \tau_{\mathcal{N}}\big[(\mathcal{J}_{\mathcal{N},\infty}\circ\Theta^j)(f)^*\big(\Theta^j_{\mathcal{N}}\circ\mathcal{J}_{\mathcal{N},\infty}\big)(f)\big]$$
$$= \mathcal{N}\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{y})\overline{f(\boldsymbol{y})}f(T^j(\boldsymbol{x}))\big|\big\langle C_{\mathcal{N}}(\boldsymbol{x}),\,W^j_{T,N}C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2,$$

and that $\tau_{\mathcal{N}}[\mathcal{J}_{\mathcal{N},\infty}(f)^*\mathcal{J}_{\mathcal{N},\infty}(f)] \longrightarrow \omega_\mu(|f|^2)$ for large N; so now the strategy is to prove that also $I_{j,N}(f)$ goes to $\omega_\mu(|f|^2) = \int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})|f(\boldsymbol{x})|^2$ when $j, N \to \infty$ with $\Gamma_T(j) < \frac{\log N}{\gamma}$. We want to prove that the difference

$$\left|I_{j,N}(f) - \int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{y})|f(\boldsymbol{y})|^2\right|$$
$$= \left|\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{y})\overline{f(\boldsymbol{y})}(f(T^j(\boldsymbol{x})) - f(\boldsymbol{y}))\mathcal{N}\big|\big\langle C_{\mathcal{N}}(\boldsymbol{x}),\,W^j_{T,N}C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2\right|$$

is negligible for large N: selecting a ball $B(T^j(\boldsymbol{x}), d_0)$, one derives

$$\leqslant \left|\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})\int_{B(T^j(\boldsymbol{x}),d_0)}\mu(\mathrm{d}\boldsymbol{y})\overline{f(\boldsymbol{y})}(f(T^j(\boldsymbol{x})) - f(\boldsymbol{y}))\mathcal{N}\big|\big\langle C_{\mathcal{N}}(\boldsymbol{x}),\,W^j_{T,N}C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2\right|$$
$$+\left|\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})\int_{\mathbb{T}^2\setminus B(T^j(\boldsymbol{x}),d_0)}\mu(\mathrm{d}\boldsymbol{y})\overline{f(\boldsymbol{y})}(f(T^j(\boldsymbol{x}))\right.$$
$$\left. - f(\boldsymbol{y}))\mathcal{N}\big|\big\langle C_{\mathcal{N}}(\boldsymbol{x}),\,W^j_{T,N}C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2\right|.$$

Applying the mean value theorem in the first double integral, we get that $\exists \boldsymbol{c} \in B(T^j(\boldsymbol{x}), d_0)$ such that

$$\left|I_{j,N}(f) - \int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{y})|f(\boldsymbol{y})|^2\right| \leqslant \int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})|\overline{f(\boldsymbol{c})}(f(T^j(\boldsymbol{x})) - f(\boldsymbol{c}))|$$
$$\times\int_{B(T^j(\boldsymbol{x}),d_0)}\mu(\mathrm{d}\boldsymbol{y})\mathcal{N}\big|\big\langle\big(W^*_{T,N}\big)^jC_{\mathcal{N}}(\boldsymbol{x}),\,C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2$$
$$+2\|f\|_0^2\int_{\mathbb{T}^2}\mu(\mathrm{d}\boldsymbol{x})\int_{\mathbb{T}^2\setminus B(T^j(\boldsymbol{x}),d_0)}\mu(\mathrm{d}\boldsymbol{y})\mathcal{N}\big|\big\langle C_{\mathcal{N}}(\boldsymbol{x}),\,W^j_{T,N}C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2,$$

where we used the uniform norm $\|\cdot\|_0$, indeed $f \in \mathcal{C}^0(\mathbb{T}^2)$. Finally, using completeness and normalization (properties 2.1), we arrive at the upper bound

$$\leqslant\|f\|_0\sup_{\substack{\boldsymbol{z}\in\mathbb{T}^2\\ \boldsymbol{c}\in B(\boldsymbol{z},d_0)}}|(f(\boldsymbol{z}) - f(\boldsymbol{c}))| + 2\|f\|_0^2\,\mathcal{N}\sup_{\substack{\boldsymbol{x}\in\mathbb{T}^2\\ \boldsymbol{y}\notin B(T^j(\boldsymbol{x}),d_0)}}\big|\big\langle C_{\mathcal{N}}(\boldsymbol{x}),\,W^j_{T,N}C_{\mathcal{N}}(\boldsymbol{y})\big\rangle\big|^2.$$

By uniform continuity, the first term can be made arbitrarily small, provided we choose d_0 small enough. For the second integral, we use theorem 2, which provides us with $N_0 = N_0(\gamma, d_0)$ depending on the same d_0, such that the second term vanishes for all $N > N_0$ and for all j such that $\Gamma_T(j) < \frac{\log N}{\gamma}$. □

5. Dynamical entropy on discrete systems

Dealing with hyperbolic systems, one expects the instability proper to the presence of a positive Lyapunov exponent to correspond to some degree of unpredictability of the dynamics: classically, the metric entropy of Kolmogorov–Sinai provides the link [27].

5.1. A classical one: Kolmogorov–Sinai metric entropy

For continuous classical systems $(\mathcal{X}, \mu, T)$ such as those introduced in section 2, the construction of the dynamical entropy of Kolmogorov–Sinai is based on subdividing $\mathcal{X}$ into measurable disjoint subsets $\{E_\ell\}_{\ell=1,2,\ldots,D}$ such that $\bigcup_\ell E_\ell = \mathcal{X}$ which form finite partitions (coarse graining *s*) $\mathcal{E}$.

Under the action of dynamical maps T in (1), any given partition $\mathcal{E}$ evolves into $T^{-j}(\mathcal{E})$ with atoms $T^{-j}(E_\ell) = \{\boldsymbol{x} \in \mathcal{X} : T^j(\boldsymbol{x}) \in E_\ell\}$; one can then form finer partitions $\mathcal{E}_{[0,n-1]} := \bigvee_{j=0}^{n-1} T^j(\mathcal{E})$ whose atoms $E_{i_0 i_1 \cdots i_{n-1}} := \bigcap_{j=0}^{n-1} T^{-j} E_{i_j}$ have volumes $\mu_{i_0 i_1 \cdots i_{n-1}} := \mu\big(E_{i_0 i_1 \cdots i_{n-1}}\big)$.

Definition 5.1.

(1) We shall set $\boldsymbol{i} = \{i_0 i_1 \cdots i_{n-1}\}$ and denote by Ω_D^n the set of D^n n-tuples with i_j taking values in $\{1, 2, \ldots, D\}$.

(2) The symbol $\hat{\boldsymbol{\imath}}$ will indicate the string $\hat{\boldsymbol{\imath}} := \{i_{n-1} i_{n-2} \cdots i_1 i_0\} \in \Omega_D^n$; the two strings $\boldsymbol{i}$ and $\hat{\boldsymbol{\imath}}$ are related by $i_j = \hat{\imath}_{n-1-j}, \forall j \in \{0, \ldots, n-1\}$.

The atoms of the partitions $\mathcal{E}_{[0,n-1]}$ describe segments of trajectories up to time n encoded by the atoms of $\mathcal{E}$ that are traversed at successive times; the volumes $\mu_{\boldsymbol{i}} = \mu(E_{\boldsymbol{i}})$ correspond to probabilities for the system to belong to the atoms $E_{i_0}, E_{i_1}, \ldots, E_{i_{n-1}}$ at successive times $0 \leqslant j \leqslant n-1$. The richness in diverse trajectories, that is the degree of irregularity of the motion (as seen with the accuracy of the given coarse graining) correspond intuitively to our idea of 'complexity' and can be measured by the Shannon entropy [16] $S_\mu(\mathcal{E}_{[0,n-1]}) := -\sum_{\boldsymbol{i} \in \Omega_D^n} \mu_{\boldsymbol{i}} \log \mu_{\boldsymbol{i}}$.

In the long run, the partition $\mathcal{E}$ attributes to the dynamics an entropy per unit time step $h_\mu(T, \mathcal{E}) := \lim_{n\to\infty} \frac{1}{n} S_\mu(\mathcal{E}_{[0,n-1]})$.

This limit is well defined [7] and the 'average entropy production' $h_\mu(T, \mathcal{E})$ measures how predictable the dynamics is on the coarse-grained scale provided by the finite partition $\mathcal{E}$. To remove the dependence on $\mathcal{E}$, the KS entropy $h_\mu(T)$ of $(\mathcal{X}, \mu, T)$ is defined as the supremum over all finite measurable partitions [7, 16] $h_\mu(T) := \sup_{\mathcal{E}} h_\mu(T, \mathcal{E})$.

5.2. Dynamics and information in the quantum setting

From an algebraic point of view, the difference between a 'quantum' triplet $(\mathcal{M}, \omega, \Theta)$ describing a quantum dynamical system and classical triplets like $\big(L^\infty_\mu(\mathbb{T}^2), \omega_\mu, \Theta\big)$ of section 2 or $(\mathcal{D}_\mathcal{N}, \tau_\mathcal{N}, \Theta_\mathcal{N})$ of section 2.1 is that ω and Θ are now a Θ-invariant state, respectively an automorphism over a non-commutative (C* or von Neumann) algebra of operators $\mathcal{M}$ [11].

- In standard quantum mechanics the algebra $\mathcal{M}$ is the von Neumann algebra $B(\mathcal{H})$ of all bounded linear operators on a suitable Hilbert space $\mathcal{H}$. If $\mathcal{H}$ has finite dimension D, $\mathcal{M}$ is the algebra of $D \times D$ matrices.
- The typical states ω are density matrices ρ, namely operators with positive eigenvalues ρ_ℓ such that $\mathrm{Tr}(\rho) = \sum_\ell \rho_\ell = 1$. Given the state ρ, the mean value of any observable $X \in B(\mathcal{H})$ is given by $\rho(X) := \mathrm{Tr}(\rho X)$.
- The ρ_ℓ of the previous point are interpreted as probabilities of finding the system in the corresponding eigenstates. The uncertainty prior to the measurement is measured by the von Neumann entropy of ρ given by $H(\rho) := -\mathrm{Tr}(\rho \log \rho) = -\sum_\ell \rho_\ell \log \rho_\ell$.
- The usual dynamics on $\mathcal{M}$ is of the form $\Theta(X) = UXU^*$, where U is a unitary operator. If one has a Hamiltonian operator that generates the continuous group $U_t = \exp \mathrm{i}tH/\hbar$ then $U := U_{t=1}$ and the time evolution is discretized by considering powers U^j.

The idea behind the notion of dynamical entropy is that information can be obtained by repeatedly observing a system in the course of its time evolution. Due to the uncertainty principle, or, in other words, to non-commutativity, if observations are intended to gather information about the intrinsic dynamical properties of quantum systems, then non-commutative extensions of the KS entropy ought first to decide whether quantum disturbances produced by observations have to be taken into account or not.

Concretely, let us consider a quantum system described by a density matrix ρ acting on a Hilbert space $\mathcal{H}$. Via the wave packet reduction postulate, generic measurement processes may reasonably well be described by finite sets $\mathcal{Y} = \{y_0, y_1, \ldots, y_{D-1}\}$ of bounded operators $y_j \in \mathcal{B}(\mathcal{H})$ such that $\sum_j y_j^* y_j = \mathbb{1}$. These sets are called partitions of unity (*p.u.*, for the sake of brevity) and describe the change in the state of the system caused by the corresponding measurement process:

$$\rho \longmapsto \Gamma_{\mathcal{Y}}^*(\rho) := \sum_j y_j \rho y_j^*. \tag{45}$$

It looks rather natural to rely on partitions of unity to describe the process of collecting information through repeated observations of an evolving quantum system [18].

Our intention is now to introduce a quantum dynamical entropy [19], based on and constructed by means of CS, and apply it to our families of discretized toral automorphisms. We will show that this quantity does reduce to the Kolmogorov–Sinai invariant, but only for time scales bounded by the logarithm of the discretization parameter N.

It is worth mentioning that the same result has been proved in [14] for two different quantum dynamical entropies (called ALF and CNT entropy) applied to finite-dimensional quantum counterparts of the hyperbolic family of UMG that we have considered within this paper. The only hypothesis used in [14] to get the above-mentioned result consisted of a dynamical localization property analogous to the one we proved in theorem 2.

As a consequence, the same results as [14], that is the convergence of ALF and CNT entropy to the KS one, can also be obtained in the present framework.

5.3. CS-quantum entropies

In order to make the description of a quantum system closer to that of a classical one, the most useful tool consists in using CS. The quantum measurement process itself can be depicted in terms of CS in such a way that the classical property can be recovered in the semiclassical limit.

Let $(\mathcal{M}, \omega, \Theta)$ be a (finite-dimensional) quantum dynamical system such as the ones introduced in section 5.2, with $\mathcal{N}$ denoting the dimension of its Hilbert space $\mathcal{H}$ and $(\mathcal{X}, \mu, T)$ its classical counterpart, the latter endowed with a classical partition $\mathcal{E} = \{E_\ell\}_{\ell=1,2,\ldots,D}$ on it (see section 5.1). Introduce in such a system a family of coherent states endowed with properties 2.1.

The map

$$\mathcal{I}(C)(\rho) := \mathcal{N} \int_C |C_{\mathcal{N}}(\boldsymbol{x})\rangle\langle C_{\mathcal{N}}(\boldsymbol{x})|\rho|C_{\mathcal{N}}(\boldsymbol{x})\rangle\langle C_{\mathcal{N}}(\boldsymbol{x})|\mu(\mathrm{d}\boldsymbol{x}), \tag{46}$$

for a measurable subset $C \subset \mathcal{X}$ and an operator ρ, is called an *instrument* [19]. The map $\rho \longmapsto \mathcal{I}(C)(\rho)$ describes the change in the state ρ of the system caused by a C-dependent measurement process (compare with (45)).

If we take the expectation of $\mathcal{I}(C)(\rho)$, that is $\mu^{(\rho)}(C) := \omega[\mathcal{I}(C)(\rho)]$, we get the probability that a measurement of the system by the instrument (46) gives values in C, when the pre-measurement state is ρ. If we wonder what is the probability that several measures, taken stroboscopically at times $t_0 = 0, t_1 = 1, \ldots, t_{n-1} = n-1$, give values in $E_{i_0}, E_{i_1}, \ldots, E_{i_{n-1}}$, we have to compose the instrument action (46) with the temporal evolution depicted in section 5.2, obtaining

$$\begin{aligned}\mathcal{P}^{\mathsf{CS}}_{i_0,i_1,\ldots,i_{n-1}} &:= \mu^{(\rho)}_{t_0,t_1,\ldots,t_{n-1}}\left(E_{i_0} \times E_{i_1} \times \cdots \times E_{i_{n-1}}\right)\\ &= \omega\big[\mathcal{I}\left(E_{i_{n-1}}\right) \circ \Theta \circ \mathcal{I}\left(E_{i_{n-2}}\right) \circ \Theta \circ \cdots \circ \mathcal{I}\left(E_{i_1}\right) \circ \Theta \circ \mathcal{I}\left(E_{i_0}\right)(\rho)\big].\end{aligned} \tag{47}$$

Using in (47) the expression for the dynamical evolution $\Theta(X) = UXU^*$ together with (46), and replacing the expectation ω with the trace (see section 5.2), we obtain

$$\begin{aligned}\mathcal{P}^{\mathsf{CS}}_{\boldsymbol{i}} = \mathcal{P}^{\mathsf{CS}}_{i_0,i_1,\ldots,i_{n-1}} &= \mathcal{N}^n \int_{E_{i_0}} \int_{E_{i_1}} \cdots \int_{E_{i_{n-1}}} \langle C_{\mathcal{N}}(\boldsymbol{x}_0)|\rho|C_{\mathcal{N}}(\boldsymbol{x}_0)\rangle\\ &\quad\times \prod_{j=1}^{n-1}[|\langle C_{\mathcal{N}}(\boldsymbol{x}_j)|U|C_{\mathcal{N}}(\boldsymbol{x}_{j-1})\rangle|^2]\mu(\mathrm{d}\boldsymbol{x}_0)\mu(\mathrm{d}\boldsymbol{x}_1)\cdots\mu(\mathrm{d}\boldsymbol{x}_{n-1}),\end{aligned} \tag{48}$$

where we have used the normalization property for the state $|C_{\mathcal{N}}(\boldsymbol{x}_{n-1})\rangle$ and the notation given in definition 5.1 for the strings $\boldsymbol{i}$.

These quantities can be seen as quantum analogue to the classical probability $\mu_{\boldsymbol{i}}$ of section 5.1 (in particular they sum up to 1) and thus can be used in computing a Shannon entropy, depending on the given dynamics U, the instrument (46), the classical partition $\mathcal{E}$, the initial state ρ and the considered time of measuring n, whose expression is

$$S(U, \mathcal{I}, \mathcal{E}, \rho, n) := -\sum_{\boldsymbol{i}\in\Omega_D^n} \mathcal{P}^{\mathsf{CS}}_{\boldsymbol{i}} \log \mathcal{P}^{\mathsf{CS}}_{\boldsymbol{i}} \quad . \tag{49}$$

The CS *quantum entropy* [19] is defined as the 'average production' in the long run of the last quantity

$$H(U, \mathcal{I}, \mathcal{E}, \rho) := \lim_{n\to\infty} \frac{1}{n} S(U, \mathcal{I}, \mathcal{E}, \rho, n) \tag{50}$$

and it is decomposable into two components. The first, called *measurement* CS *quantum entropy*, is independent of the dynamics, originated by the pure measurement process, and obtained by replacing the unitary operator U in (50) with the identity on $\mathcal{H}$; its expression is

$$H_{\mathrm{meas}}(\mathcal{I}, \mathcal{E}, \rho) := H(\mathbb{1}_{\mathcal{N}}, \mathcal{I}, \mathcal{E}, \rho). \tag{51}$$

The second amount is the remaining part

$$H_{\mathrm{dyn}}(U, \mathcal{I}, \mathcal{E}, \rho) = H(U, \mathcal{I}, \mathcal{E}, \rho) - H_{\mathrm{meas}}(\mathcal{I}, \mathcal{E}, \rho) \tag{52}$$

and is supposed to incorporate the dynamic dependence.

5.4. CS entropies for discrete classical systems

The quantum entropy of the last section can be seen as an algebraic quantity, and needs nothing more than the algebraic framework already developed in sections 2–4, in order to be defined. In particular, we are going to estimate the CS entropy of discrete classical systems $(\mathcal{D}_{\mathcal{N}}, \tau_{\mathcal{N}}, \Theta_{\mathcal{N}})$ using the lattice states of definition 2.1.

Theorem 3. *Let* $(\mathbb{T}^2, \mu, T)$ *be the classical dynamical system of section 2, which is the continuous limit of a sequence of finite-dimensional discrete dynamical systems* $(\mathcal{D}_{\mathcal{N}}, \tau_{\mathcal{N}}, \Theta_{\mathcal{N}})$. *If*

(1) $W_{T,N}$ *is the unitary evolution operator of (27);*
(2) $\mathcal{I}$ *in the instrument (46) constructed with the LS of definition 2.1;*

(3) $\mathcal{E} = \{E_0, E_1, \dots, E_{D-1}\}$ *is a finite measurable partition of* $\mathbb{T}^2$*;*
(4) ρ *is the tracial state* $\frac{1}{\mathcal{N}}\mathbb{1}_{\mathcal{N}}$*;*

then there exists an α *such that*

$$\lim_{\substack{n,N\to\infty\\ n<\alpha\log N}} \frac{1}{n}|S(W_{T,N},\mathcal{I},\mathcal{E},\rho,n) - S_\mu(\mathcal{E}_{[0,n-1]})| = 0.$$

In order to prove theorem 3, we need the following auxiliary result.

Lemma 5.1. *Suppose we have a sequence* $\{g_N\}$ *of* $L^2_\mu(\mathbb{T}^2)$ *functions such that* $\|g_N\|_2 \leqslant 1$*,* $\forall N \in \mathbb{N}^+$ *(*$\|\cdot\|_2$ *meaning the* $L^2_\mu(\mathbb{T}^2)$*-norm).*

Using the quantities $K_{N,n}(\boldsymbol{x},\boldsymbol{y})$ *of definition 4.2 we have that, for any given A and B measurable subsets of* $\mathbb{T}^2$ *and N large enough, it holds*

$$R_N := \left| \int_B \mu(\mathrm{d}\boldsymbol{x}) g_N(\boldsymbol{x}) \mathcal{N} \int_A \mu(\mathrm{d}\boldsymbol{y}) |K_{N,1}(\boldsymbol{x},\boldsymbol{y})|^2 - \int_{B\cap T^{-1}(A)} \mu(\mathrm{d}\boldsymbol{x}) g_N(\boldsymbol{x}) \right| \leqslant \varepsilon_B(N),$$

where $\varepsilon_B(N) \longrightarrow 0$ *with* $N \longrightarrow \infty$*.*

The symbol ε_B does not imply any dependence of the bounding term ε_B on the subset B; it is just a way of writing that will be of use in the following.

Proof of lemma 5.1. Resorting to the use of the characteristic functions $\mathcal{X}_A$ and $\mathcal{X}_B$, using triangular inequality and collecting terms, R_N can be rewritten as

$$\begin{aligned} R_N &\leqslant \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{x}) |\mathcal{X}_B(\boldsymbol{x}) g_N(\boldsymbol{x})| \cdot \left| \mathcal{N} \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{y}) \mathcal{X}_A(\boldsymbol{y}) \left| K_{N,1}(\boldsymbol{x},\boldsymbol{y}) \right|^2 - \mathcal{X}_{T^{-1}(A)}(\boldsymbol{x}) \right| \\ &= \left\| \mathcal{X}_B g_N \left[\mathcal{N} \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{y}) \mathcal{X}_A(\boldsymbol{y}) |K_{N,1}(\cdot,\boldsymbol{y})|^2 - \mathcal{X}_A(T(\cdot)) \right] \right\|_1, \end{aligned}$$

and using the Cauchy–Schwartz inequality

$$\leqslant \|\mathcal{X}_B g_N\|_2 \cdot \left\| \mathcal{N} \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{y}) \mathcal{X}_A(\boldsymbol{y}) |K_{N,1}(\cdot,\boldsymbol{y})|^2 - \mathcal{X}_A(T(\cdot)) \right\|_2. \tag{53}$$

Now we use the hypothesis, so that

$$\|\mathcal{X}_B g_N\|_2^2 = \int_B |g_N(\boldsymbol{x})|^2 \mu(\mathrm{d}\boldsymbol{y}) \leqslant \|g_N\|_2^2 \leqslant 1. \tag{54}$$

Putting together (53) and (54) and using proposition 4.1 (with $f = \mathcal{X}_A$ and $n = 1$), we get the result. □

We are now in position to conclude with:

Proof of theorem 3. Let us start to compute the expectation $\mathcal{P}_i^{\mathsf{CS}}$. In terms of the quantity introduced in points (1)–(4) of the statement, equation (48) can be rewritten as

$$\begin{aligned} \mathcal{P}_i^{\mathsf{CS}} = \mathcal{N}^{n-1} \int_{E_{i_0}} \int_{E_{i_1}} \cdots \int_{E_{i_{n-1}}} &\langle C_{\mathcal{N}}(\boldsymbol{x}_0) | \mathbb{1}_{\mathcal{N}} | C_{\mathcal{N}}(\boldsymbol{x}_0) \rangle \\ &\times \prod_{j=1}^{n-1} [|\langle C_{\mathcal{N}}(\boldsymbol{x}_j) | W_{T,N} | C_{\mathcal{N}}(\boldsymbol{x}_{j-1}) \rangle|^2] \mu(\mathrm{d}\boldsymbol{x}_0) \mu(\mathrm{d}\boldsymbol{x}_1) \cdots \mu(\mathrm{d}\boldsymbol{x}_{n-1}) \end{aligned}$$

and using the normalization property for the state $|C_{\mathcal{N}}(\boldsymbol{x}_0)\rangle$ and resorting to definition 4.2

$$= \int_{E_{i_{n-1}}} \cdots \int_{E_{i_1}} \int_{E_{i_0}} \mu(\mathrm{d}\boldsymbol{x}_{n-1}) \times \prod_{j=1}^{n-1} [\mathcal{N}|K_{N,1}(\boldsymbol{x}_j, \boldsymbol{x}_{j-1})|^2 \mu(\mathrm{d}\boldsymbol{x}_{j-1})]. \tag{55}$$

Now we start an iteratation procedure, consisting of two points.

(1) consider the function

$$g_N(\boldsymbol{x}_1) := \int_{E_{i_{n-1}}} \cdots \int_{E_{i_3}} \int_{E_{i_2}} \prod_{j=2}^{n-1} [\mathcal{N}|K_{N,1}(\boldsymbol{x}_j, \boldsymbol{x}_{j-1})|^2 \mu(\mathrm{d}\boldsymbol{x}_j)], \tag{56}$$

all the factors inside the integrals of (56) are positive, so that extending the integration domain and giving the form of $K_{N,1}(\boldsymbol{x}_j, \boldsymbol{x}_{j-1})$ explicitly, we get the bound

$$g_N(\boldsymbol{x}_1) \leqslant \int_{\mathbb{T}^2} \cdots \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \prod_{j=2}^{n-1} [\mathcal{N}|\langle C_{\mathcal{N}}(\boldsymbol{x}_j), W_{T,N} C_{\mathcal{N}}(\boldsymbol{x}_{j-1})\rangle|^2 \mu(\mathrm{d}\boldsymbol{x}_j)] = 1$$

from completeness and normalization, so that it follows $\|g_N\|_2 \leqslant 1$.

(2) By means of (56), equation (55) can be rewritten as

$$\mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} = \int_{E_{i_1}} \mu(\mathrm{d}\boldsymbol{x}_1) g_N(\boldsymbol{x}_1) \mathcal{N} \int_{E_{i_0}} \mu\,(\mathrm{d}\boldsymbol{x}_0)\, |K_{N,1}(\boldsymbol{x}_1, \boldsymbol{x}_0)|^2.$$

Now lemma 5.1 guarantees that there exists a positive sequence $\varepsilon_{E_{i_1}}(N)$ such that

$$\left| \mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} - \int_{E_{i_1} \cap T^{-1}(E_{i_0})} \mu(\mathrm{d}\boldsymbol{x}_1) g_N(\boldsymbol{x}_1) \right| \leqslant \varepsilon_{E_{i_1}}(N),$$

with $\varepsilon_{E_{i_1}}(N) \longrightarrow 0$ for $N \longrightarrow \infty$. By iterating this procedure $(n-1)$ times (consisting in isolating a single $K_{N,1}(\boldsymbol{x}_j, \boldsymbol{x}_{j-1})$ and grouping all the others in a single bounded function $g_N(\boldsymbol{x}_j)$) and using the triangle inequality for $|\cdot|$, we finally arrive at the result

$$\left|\mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} - \mu\left(E_{i_{n-1}} \cap T^{-1}\left(E_{i_{n-2}}\right) \cap \cdots \cap T^{1-n}\left(E_{i_0}\right)\right)\right| = \left|\mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} - \mu_{\hat{\boldsymbol{\imath}}}\right| \leqslant \varepsilon(N),$$

with

$$\varepsilon(N) := \sum_{\ell=1}^{n-1} \varepsilon_{E_{i_\ell}}(N) \longrightarrow 0 \qquad \text{for} \quad N \longrightarrow \infty, \tag{57}$$

$\mu_{\boldsymbol{j}}$ meaning the classical probability of section 4.1 and $\hat{\boldsymbol{\imath}}$ denoting the string $\boldsymbol{i}$ reversed, as in definition 5.1.2.

We now define two *density matrices*, with the aim of computing their *von Neumann Entropy* (see section 5.2), that are both diagonal in the basis $\{|\boldsymbol{i}\rangle\}_{\boldsymbol{i}\in\Omega_D^n}$ of the D^n-dimensional Hilbert space $\mathcal{H}_{D^n}$:

$$\rho := \sum_{\boldsymbol{i}\in\Omega_D^n} \mu_{\boldsymbol{i}} |\boldsymbol{i}\rangle\langle\boldsymbol{i}|, \qquad \sigma := \sum_{\boldsymbol{i}\in\Omega_D^n} \mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} |\boldsymbol{i}\rangle\langle\boldsymbol{i}|.$$

Resorting to the *trace norm* $\|A\|_1 := \mathrm{Tr}|\,A\,| = \mathrm{Tr}\sqrt{A^\dagger A}$, we use (57) to estimate $\|\rho - \sigma\|_1$, that is

$$\Delta(n) := \|\rho - \sigma\|_1 \leqslant D^n \varepsilon(N).$$

Finally, by the continuity of the von Neumann entropy [29], we get

$$|H(\rho) - H(\sigma)| \leqslant \Delta(n) \log D^n + \eta(\Delta(n)),$$

that is $|S(W_{T,N}, \mathcal{I}, \mathcal{E}, \rho, n) - S_\mu(\mathcal{E}_{[0,n-1]})| \leqslant \Delta(n) \log D^n + \eta(\Delta(n))$; indeed the two von Neumann entropies $H(\rho)$ and $H(\sigma)$ are nothing but the Shannon entropy of the refinements $\mathcal{E}_{[0,n-1]}$ of the classical partition (see section 4.1), respectively the Shannon entropy (49) leading to the CS-quantum entropy.

Since, from $n \leqslant \alpha \log N$, $D^n \leqslant N^{\alpha \log D}$, if we want the bound $D^n \varepsilon(N)$ to converge to zero with $N \longrightarrow \infty$, the parameter α has to be chosen accordingly. □

By means of theorem 3, a positive CS-entropy production is then associated with discrete systems whose continuous limit exhibits a positive KS-entropy production, which correspond in turn to the sum of all positive Lyapunov exponents of the continuous classical system, as stated by Pesin's theorem [15].

This positive CS-entropy production is entirely due to the dynamical component $H_{\mathrm{dyn}}(W_{T,N}, \mathcal{I}, \mathcal{E}, \rho)$ of (52), being the measurement CS entropy (51) equal to zero, as stated in the next proposition:

Proposition 5.1. *Let $\mathcal{I}$ and $\mathcal{E}$ be the instrument, respectively the finite measurable partition of the statement of theorem 3 and let ρ be the tracial state $\frac{1}{N}\mathbb{1}_{\mathcal{N}}$. There exists an α' such that*

$$\lim_{\substack{n,N\to\infty \\ n<\alpha' \log N}} \frac{1}{n} S(\mathbb{1}_{\mathcal{N}}, \mathcal{I}, \mathcal{E}, \rho, n) = 0.$$

Proof. Performing a proof completely analogous to the one for theorem 3, we find an α' such that

$$\lim_{\substack{n,N\to\infty \\ n<\alpha' \log N}} \frac{1}{n} |S(\mathbb{1}_{\mathcal{N}}, \mathcal{I}, \mathcal{E}, \rho, n) - S_\mu(\mathcal{E}'_{[0,n-1]})| = 0, \tag{58}$$

with $\mathcal{E}'_{[0,n-1]}$ now given by $\mathcal{E}'_{[0,n-1]} := \bigvee_{j=0}^{n-1} \mathbb{1}^j(\mathcal{E}) = \mathcal{E} \bigvee \mathcal{E} \bigvee \cdots \bigvee \mathcal{E}$ (see section 5.1), so that

$$S_\mu(\mathcal{E}'_{[0,n-1]}) = S_\mu(\mathcal{E}) \leqslant \log D \tag{59}$$

independent of n.

Now we use the triangular inequality together with (59), obtaining

$$\frac{1}{n} S(\mathbb{1}_{\mathcal{N}}, \mathcal{I}, \mathcal{E}, \rho, n) \leqslant \frac{1}{n} |S(\mathbb{1}_{\mathcal{N}}, \mathcal{I}, \mathcal{E}, \rho, n) - S_\mu(\mathcal{E}'_{[0,n-1]})| + \frac{\log D}{n}, \tag{60}$$

and so the result follows from (58). □

6. Conclusions

In this work, we studied the footprints of chaos present in classical dynamical systems on the two-dimensional torus after a discretization has forced these systems to move on a regular lattice of spacing $\frac{1}{N}$, with finite number of sites N^2.

Discretizing is similar to quantizing; in particular, as for the classical limit $\hbar \to 0$, we have set up a solid theoretical framework to discuss the continuous limit $N \to \infty$.

Inspired by the semiclassical analysis, we developed an algebraic discretization technique by mimicking the well-known anti-Wick schemes of quantization, in particular we made use of a family of suitably defined lattice states with properties that, in a quantum setting, are typical of coherent states.

The result is the appearance of a logarithmic time scale when the discrete hyperbolic systems tend to their continuous limit; namely, the continuous and discrete dynamics agree up to a breaking time which is proportional to the logarithm of the lattice spacing.

We also used the entropy production as a parameter of chaotic behaviour. In particular, the notion of CS-quantum entropy has been used: this reproduces the classical metric entropy of Kolmogorov and Sinai if applied to classical continuous systems.

The CS-quantum entropy does converge to the KS invariant, but on logarithmic time scales too.

Acknowledgment

The author wishes to thank Dr F Benatti for stimulating discussions and useful advice.

Appendix A. Sketch of the proofs of propositions 3.1, 3.2 and 3.3

Proof of proposition 3.1. (1) Let us start by considering matrices with positive trace, that is positive eigenvalues $\left(\lambda, \lambda^{-1}\right)$; the case of negative trace will be considered in the next point (2). In the (non-orthogonal) reference system $(\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2)$ oriented along eigenvectors $(|\boldsymbol{e}_+\rangle, |\boldsymbol{e}_-\rangle)$, the time evolution is described by

$$(c_1, c_2) \xrightarrow[n\in\mathbb{N}]{T^{\pm n}} (\lambda^{\pm n} c_1, \lambda^{\mp n} c_2), \tag{A.1}$$

thus orbits are simply given by $c_1 c_2 = \text{const}$, that in the reference system $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ reads as (11), indeed the relation between the coordinates in the two systems is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \cos\beta \\ 0 & \sin\beta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{A.2}$$

Among these orbits, we choose the two that are tangent (and so closest) to the unit ball $B_T(0)$: of course they remain tangent and closest even during evolution $B_T(0) \longmapsto B_T(n)$ and so they give us the right expression for the surrounding orbits of $B_T^{(n)}$, that is (13).

By means of (A.1) and (A.2) we have an expression for the $\pm n$-evolved unit ball, that is $B_T(n)$; among its surface's points we choose the farthest ones and we determine their norm, getting the expression for $D_T(n)$ contained in (14).

Now we use the expression $\sinh^{-1}(q) = \log(\sqrt{q^2+1} + q)$, that holds for all $q > 0$, in particular for $q = (\lambda^n - \lambda^{-n})/\sin\beta$ $(\sin\beta > 0)$, so that from (14) we get for $D_T(n)$ the expression given by (15) that shows the monotonicity in n of this function; this monotonicity, together with the definitions (10) of $B_T^{(n)}$, gives us the equivalence between $D_T^{(n)}$ and $D_T(n)$.

The linear matrix action T maps the unit ball $B_T(0)$ in the ellipse $B_T(1)$ and $D_T(1)$ is its major semi-axis; from definition 2.2, we have

$$\eta^2 = \sup_{|\boldsymbol{v}\rangle\in\mathbb{R}^2} \langle \boldsymbol{v}|T^\dagger T|\boldsymbol{v}\rangle = \sup_{|\boldsymbol{v}\rangle\in\mathbb{R}^2} \|T\big|\boldsymbol{v}\rangle\|^2_{\mathbb{R}^2} = [D_T(1)]^2,$$

so that $\eta = D_T(1)$ and (12) follows from expression (14), with $n = 1$.

Expressions in (16) can be easily deduced from (14).

(2) Let us now note that every map T, whose trace is negative, may be written as the composition of $-\mathbb{1}_2$ (the identity map) with the map $-T$, which has positive trace; the same holds true for the iterates $\{T^k\}_{k\ \text{odd}}$. Since multiplying by $-\mathbb{1}_2$ amounts to performing the transformation $(x, y) \longmapsto (-x, -y)$, both the orbits (11) and the surrounding surface (12), which exhibit

a central symmetry, remain the same for negative trace maps. The same argument can be applied to the diameter $D_T(n)$ of (14), which is invariant for coordinate reflection too. □

Proof of proposition 3.2. Let us consider matrices T with $\mathrm{Tr}\, T = 2$, that is $t = 1$, the case $t = -1$ being equivalent, as it is possible to prove in the same way as point (2) of the proof of proposition 3.1. In the orthogonal reference system $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ of the statement, the action of T^n is described by a matrix in Jordan canonical form, that is

$$\begin{pmatrix} x \\ y \end{pmatrix} \underset{T^n}{\rightarrow} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & nJ' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{A.3}$$

where $J' = t_{12} - t_{21}$, thus orbits are simply given by $y = \mathrm{const}$. In order to apply the argument of point (2) of the proof of proposition 3.1, when $t = -1$, we endow this class of orbits with a coordinate reflection symmetry, and this leads to equation (17).

Among these orbits, we choose the one that is tangent (and so closest) to the unit ball $B_T(0)$: of course it remains tangent and closest even during evolution $B_T(0) \longmapsto B_T(n)$ and so it gives us the right expression for the surrounding orbit of $B_T^{(n)}$, that is (18).

By means of (A.3) we have an expression for the $\pm n$-evolved unit ball, that is $B_T(n)$; among its surface's points we choose the farthest ones and we determine their norm, getting the expression for $D_T(n)$ contained in (20), with $J = |J'|$.

Using once more the expression $\sinh^{-1}(q) = \log(\sqrt{q^2 + 1} + q)$, that holds for all $q > 0$, in particular for $q = nJ$, from (20) we get for $D_T(n)$ the expression given by (21); using monotonicity, we get the equivalence $D_T^{(n)} = D_T(n)$.

From $\eta = D_T(1)$ (see proof of proposition 3.1), equation (19) can be obtained from expression (20), with $n = 1$.

Expressions in (22) and (23) can be easily deduced and verified from (20). □

Proof of proposition 3.3. The semi-trace t of the matrix T can only assume values in $\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$, indeed all entries of T are integer and $|t| < 1$. We read from equation (8) that $t = \cos\phi$ and so we have for ϕ the only possible values $\left\{\pm\frac{2}{3}\pi, \pm\frac{1}{2}\pi, \pm\frac{1}{3}\pi\right\}$; each of these values make the time evolution periodic, as can be deduced from equation (8). All these cases are similar; we now prove the statement for $t = \frac{1}{2}$.

$t = \frac{1}{2}$: we have $\phi = \pm\frac{1}{3}\pi$ and so we get from equation (8) that $T^3 = -\mathbb{1}_2$. The period of evolution is six and the sequence of T power is equivalent to $\mathbb{1}_2, T, -T^{-1}, -\mathbb{1}_2, -T, T^{-1}, \mathbb{1}_2$ and so on.

By using equation (9) of definition 3.2 we see that the sequence $\{B_T(n)\}_{n\in\mathbb{N}}$ of an n-evolved ball is equivalent to $B_T(0), B_T(1), B_T(-1), B_T(0), B_T(1), B_T(-1), \ldots$, thus, the sequence of diameter $\{D_T(n)\}_{n\in\mathbb{N}}$ is given by $D_T(0), D_T(1), D_T(-1), \ldots$.

As argued in the proof of proposition 3.1 (point (1)), $D_T(1) = \eta$; moreover, $D_T(-1) = \eta$ too. Indeed, as the spectra of $|T|$ consist of the two eigenvalues (η, η^{-1}), the same is true for the spectra of $|T^{-1}|$.

Using the last observation, the sequence of diameter becomes $0, \eta, \eta, 0, \eta, \eta, \ldots$ and so equations (24)–(25) hold true for the case $t = \frac{1}{2}$.

The cases $t = -\frac{1}{2}$ and $t = 0$ can be proved in a similar way. □

References

[1] Zaslavsky G M 1985 *Chaos in Dynamic Systems* (Chur: Harwood Academic)

[2] Giannoni M-J, Voros A and Zinn-Justin J (ed) 1991 *Chaos and Quantum Physics* vol 1989 *(Les Houches Session LII of Les Houches Summer School of Theoretical Physics)* (Amsterdam: North-Holland)

[3] Devaney R 1989 *An Introduction to Chaotic Dynamical Systems* (Reading, MA: Addison-Wesley)

[4] Wiggins S 1990 *Dynamical Systems and Chaos* (New York: Springer)
[5] Schuster H G 1995 *Deterministic Chaos* 3rd edn (Weinheim: VCH)
[6] Casati G and Chirikov B 1995 *Quantum Chaos Between Order and Disorder* (Cambridge: Cambridge University Press)
[7] Katok A and Hasselblatt B 1999 *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications)* (Cambridge: Cambridge University Press)
[8] Bouclet J M and De Bievre S 2004 Long time propagation and control on scarring for perturbed quantized hyperbolic toral automorphisms *Preprint* math-ph/0409069
[9] Fannjiang A, Nonnenmacher S and Wolowski L 2004 Dissipation time of quantized toral maps *Preprint* math-ph/0406055
[10] Schubert R 2004 Semiclassical behaviour of expectation values in time evolved Lagrangian states for large times *Preprint* math.MP/0402038
[11] Cappellini V 2004 Quantum dynamical entropies and complexity in dynamical systems *PhD Thesis* University of Trieste *Preprint* math-ph/0403035
[12] Bonechi F and De Bièvre S 2003 Controlling strong scarring for quantized ergodic toral automorphisms *Duke Math. J.* **3** 571–87
[13] Werner R F 1995 The classical limit of quantum theory *Preprint* quant-ph/9504016
[14] Benatti F, Cappellini V, De Cock M, Fannes M and Vanpeteghem D 2003 Classical limit of quantum dynamical entropies *Rev. Math. Phys.* **15** 847–75
[15] Mañé R 1987 *Ergodic Theory and Differentiable Dynamics* (Berlin: Springer)
[16] Alekseev V M and Yakobson M V 1981 Symbolic dynamics and hyperbolic dynamical systems *Phys. Rep.* **75** 287–325
[17] Connes A, Narnhofer H and Thirring W 1987 Dynamic entropy of C*-algebras and Von Neumann algebras *Commun. Math. Phys.* **112** 691–719
[18] Alicki R and Fannes M 1994 Defining quantum dynamical entropy *Lett. Math. Phys.* **32** 75–82
[19] Słomczyński W and Życzkowski K 1994 Quantum chaos: an entropy approach *J. Math. Phys.* **35** 5674–700
[20] Benatti F, Cappellini V and Zertuche F 2004 Quantum dynamical entropies in discrete classical chaos *J. Phys. A: Math. Gen.* **37** 105–30
[21] Benatti F and Cappellini V 2005 Continuous limit of discrete sawtooth maps and its algebraic framework *J. Math. Phys.* **46** 062702
[22] Bratteli O and Robinson D W 1979 *Operator Algebras and Quantum Statistical Mechanics (vol 1: C*- and W*-Algebras, Symmetry Groups, Decomposition of States) (Texts and Monographs in Physics)* (New York: Springer)
[23] Hewitt E and Stromberg K 1969 *Real and Abstract Analysis* 2nd edn (Berlin: Springer)
[24] Rudin W 1987 *Real and Complex Analysis* 3rd edn (New York: McGraw-Hill)
[25] Marklof J and Rudnick Z 1999 Quantum unique ergodicity for parabolic maps *Preprint* math-ph/9901001
[26] Riesz F and Sz.-Nagy B 1955 *Functional Analysis* (New York: Ungar)
[27] Ford J and Ilg M 1992 Eigenfunctions, eigenvalues, and time evolution of finite, bounded, undriven, quantum systems are not chaotic *Phys. Rev.* A **45** 6165–73
[28] Narnhofer H 1992 Quantized Arnold cat maps can be entropic K-systems *J. Math. Phys.* **33** 1502–10
[29] Fannes M 1973 A continuity property of the entropy density for spin lattice systems *Commun. Math. Phys.* **31** 291–4